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Mathematical Research Statement

I am a superalgebraist: The objects I study arise from considerations in physics, such as supersymmetry and quantum-statistical mechanics, and natural generalizations in the theory of non-associative and associative algebras. Namely, I investigate the representation theory of Lie superalgebras, which are defined by a non-associative operation, using both the structure and representation theory of associative superalgebras. My research in supermathematics is key in the development of the infinite-dimensional representation theory of orthosymplectic Lie superalgebras and applications of the super analogue of reduction algebras. In particular, I have contributed to the body of scholarship on the hard problem of realizing explicit bases of infinite-dimensional representations of orthosymplectic Lie superalgebras; and, I have extended the literature on diagonal reduction algebras to diagonal reduction superalgebras. Additionally, I study combinatorial objects termed generalized parking functions using enumerative and algebraic techniques.

The subsequent sections[1] will have the following explanatory layout of subsections to begin a more detailed discussion on the goals and outcomes of my research program:

i) Historical background of research questions

ii) Results

iii) Open problems and future work

I Lie superalgebras and representation theory

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a super vector space over the complex field $\mathbb{C}$. This means $\mathfrak{g}$ is not ony $\mathbb{Z}_2$-graded but an object of the symmetric monoidal category $\mathbf{SVect}_{\mathbb{C}}$ with its twisted braiding. Elements of $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$, and the subspaces themselves, are called even and odd, respectively. The parity of a non-zero homogeneous element is expressed through the parity function $|x| \mapsto \bar{i} \in \mathbb{Z}_2$, $x \in \mathfrak{g}_{\bar{i}}$, while zero has either parity.

Given a bilinear, parity-respecting product/bracket $[\cdot,\cdot]$ on $\mathfrak{g}$, the pair $(\mathfrak{g}, [\cdot,\cdot])$, and $\mathfrak{g}$ itself, is called a *Lie superalgebra* whenever $[\cdot,\cdot]$ is skew-supersymmetric and satisfies the super Jacobi identity. That is, $[\cdot,\cdot]\big|_{\mathfrak{g}_{\bar{0}}}$ and $[\cdot,\cdot]\big|_{\mathfrak{g}_{\bar{0}}\times\mathfrak{g}_{\bar{1}}}$ are skew-symmetric; $[\cdot,\cdot]\big|_{\mathfrak{g}_{\bar{1}}}$ is symmetric; and, $0 = (-1)^{|x||z|}[x,[y,z]] + (-1)^{|z||y|}[z,[x,y]] + (-1)^{|y||x|}[y,[z,x]]$. Any Lie algebra can be made a Lie superalgebra by giving a decomposition where the odd space is the zero space. For a super vector space V, we form the general linear Lie algebra $\mathfrak{gl}(V)$ as the direct sum of parity-preserving and parity-reversing endomorphisms on V with the supercommutator as the bracket.

[1]Because of the sensitive nature of submitting results and planning future research experiences with undergraduate students, I will omit quite a few details from Section III.

A certain complex Lie subsuperalgebra of $\mathfrak{gl}(\mathbb{C}^{1|2n})$ is the *orthosymplectic Lie superalgebra* $\mathfrak{osp}(1|2n)$ of transformations on $\mathbb{C}^{1|2n}$ preserving an even, non-degenerate, supersymmetric, bilinear form—this form is unique up to a change of basis as the ground field is $\mathbb{C}$. Infinite-dimensional representations of $\mathfrak{osp}(1|2n)$ are bracket- and parity-preserving maps from $\mathfrak{osp}(1|2n)$ into $\mathfrak{gl}(V)$ for some infinite-dimensional (as a non-graded vector space) V. See [7, 41] for a general treatment of Lie superalgebras and their representations.

Question I.1 (Classification)**.** *Can we classify all irreducible infinite-dimensional representations of* $\mathfrak{osp}(1|2n)$*. Physics perspective: Can we classify all infinite-dimensional unitary irreducible representations (knowing that none of them are finite-dimensional)?*

Question I.2 (Matrix elements)**.** *For an infinite-dimensional (unitary) irreducible representations* V *of* $\mathfrak{osp}(1|2n)$*, can we provide an explicit action on a basis of* V *by the generators of* $\mathfrak{osp}(1|2n)$ *(or can we find matrix elements)?*

i) Historical background of research questions

The first recorded example of a Lie superalgebra that is not also a Lie algebra is from 1941's [48] in which Whitehead defined a product on the graded homotopy groups of a pointed topological space. Then interest in superalgebras from mathematicians [29, see the references therein] and physicists [11, for a history] alike grew in the 1970s, culminating with Kac's classification of simple finite-dimensional Lie superalgebras over algebraically closed fields of characteristic zero [28] and foundational steps[27] in the classification of their representations. Around the same time, all finite-dimensional representations of $\mathfrak{osp}(1|2n)$ were classified and found to be completely reducible [42, for more], a nice similarity to the case of semi-simple Lie algebras made more remarkable by the Djokovic-Hochshild Theorem (see [10] and prequel).

Hughes [26] began the systematic study of infinite-dimensional representations of $\mathfrak{osp}(1|2)$, whose universal enveloping algebra $U(\mathfrak{osp}(1|2))$ is the so-called dispin algebra. More generally, with the influence of Green [18] and others, Ganchev and Palev showed that the odd generators for $\mathfrak{osp}(1|2n)$ also satisfy the paraboson triple relations for n pairs of paraboson operators [15]; these paraboson operators generate the paraboson algebra which is isomorphic to $U(\mathfrak{osp}(1|2n))$. In terms of $\mathfrak{osp}(1|2n)$-representation theory, this means that the paraboson Fock space $\mathcal{H}_n(p)$ is an infinite-dimensional unitary irreducible representation [37] for each positive integer p, giving a partial answer to Question I.1. The previous reference and recent work [16] provide partial answers to Question I.2. Coulembier [8, 9, Theorem 8 is corrected in the former] and others have also considered infinite-dimensional representations of orthosymplectic Lie superalgebras using tensor product constructions in the vein of the semi-simple Lie algebra case[5] of Bernstein and Gelfand. My research has produced new directions differing from the paraboson Fock spaces, adding to the answer of Question I.1 and making explicit bases out of the theoretical decomposition results of Coulembier and Ferguson [14], a partial solution to Question I.2.

ii) Results: Bases of new infinite-dimensional representations

In my dissertation [49], I provided a complete decomposition, for all $n > 1$, of the infinite-dimensional tensor prodcut representation $\mathbb{C}[x_1, x_2, \ldots, x_n] \otimes \mathbb{C}^{1|2n}$ into irreducible summands isomorphic to $\mathbb{C}[x_1, x_2, \ldots, x_n] \otimes \mathbb{C}$ and $\mathbb{C}[x_1, x_2, \ldots, x_n] \otimes \mathbb{C}^{2n}$, respectively. While the first summand is isomorphic to the (paraboson) Fock space $\mathcal{H}_n(1)$ above, the second summand is a new infinite-dimensional (unitary) irreducible representation not isomorphic to any of the paraboson Fock spaces. This result not only addresses Question I.1 but Question I.2, too, as I found an explicit basis for the representation space in terms of the odd generators acting as special differential operators. Furthermore, I extended this result by introducing a new family of infinite-dimensional representations.

The techniques I used include defining intertwining operators for the decomposition problem and defining a new superalgebra homomorphism from $U(\mathfrak{osp}(1|2n))$ to the Weyl-Clifford symmetric algebra $A^{-}_{n|2n}$ of [23], which means I proved that any infinite-dimensional representation of $A^{-}_{n|2n}$, such as certain Clifford-valued function spaces, can be viewed as an infinite-dimensional representation of $\mathfrak{osp}(1|2n)$.

iii) Future: Decomposing other tensor product representations

The initial research questions are difficult problems and have many different directions, some of which I am continuing in collaboration with Dimitar Grantcharov, my PhD advisor, and some of which I am continuing individually. Other questions I have posed having PhD students in mind to whom I can present potential dissertation topics that will allow them to contribute significantly to the field while completing their degree under my supervision.

1. What is the decomposition of $\mathbb{C}[x] \otimes \mathbb{C}^{1|2}$ that is not answered in [49]completely? There are three summands (from three singular/primitive vectors) instead of two. There is room to relate the answer to this question to the representation theory of the diagonal reduction superalgebra described later in this document.

2. What is the solution to the decomposition problem for $\mathcal{H}_n(p) \otimes \mathbb{C}^{1|2n}$? Furthermore, how might one extend the results of [49] to exhibit bases of the summands?

II Reduction (super)algebras

Another important algebra in literature goes by many names: Mickelsson algebras, step algebras, Zhelobenko algebras, transvector algebras, and translator algebras. Among the many usages of this is algebra are its representations as linear operators on the solution spaces of various physical equations [51], including the Laplacian. For example, fix $n \geq 3$, and let A_n be the nth Weyl algebra over $\mathbb{R}$ generated by the elements $\{D_1, D_2, \ldots, D_n, M_1, M_2, \ldots, M_n\}$ with the following relations:

$$[D_i, D_j] = [M_i, M_j] = [D_i, M_j] - \delta_{ij} = 0,$$

where $[\cdot,\cdot]$ is the usual commutator given by $[A,B]=AB-BA$. Then the space of polynomials $V=\mathbb{R}[x_1,x_2,\dots,x_n]$ is an irreducible A_n-module through the representation of A_n as differential operators with polynomial coefficients (how A_n is often defined). Explicitly, D_i acts as partial differentiation with respect to x_i, written as ∂_{x_i}, and M_i acts as multiplication by x_i.

Now consider the Laplacain $\Delta=\sum_{i=1}^{n}\partial_{x_i}^2$ and the Euler operator $\mathbb{E}=\sum_{i=1}^{n}x_i\partial_{x_i}$. The set $\left\{-\frac{1}{2}\Delta, -\left(\mathbb{E}+\frac{n}{2}\right), \frac{1}{2}\sum_{i=1}^{n}x_i^2\right\}$ forms an $\mathfrak{sl}(2)$-triple and induces an algebra map $U(\mathfrak{sl}(2))\hookrightarrow A_n$. We have a system of equations $Jv=0$, where J is the left ideal of A_n generated by Δ with solutions $v\in V$. Denoting the solution space by A^{Δ}, we see A^{Δ} coincides with the space of singular/primitive vectors V^+ in the $\mathfrak{sl}(2)$-module V (hence a module of $U(\mathfrak{sl}(2))$); namely, V^+ is the space of harmonic polynomials in $\mathbb{R}[x_1,x_2,\dots,x_n]$. Furthermore, if $N=N_{A_n}(J)$ is the largest subalgebra of A_n in which J is a two-sided ideal (N is also called the normalizer of J), then the space of harmonic polynomials in $\mathbb{R}[x_1,x_2,\dots,x_n]$ is an invariant subspace for the action of the quotient algebra N/J. This algebra N/J is the *Mickelsson algebra* for $\mathfrak{sl}(2)$ by the map $\mathfrak{sl}(2)\hookrightarrow U(\mathfrak{sl}(2))\hookrightarrow A_n$. In summary, we have shown that the representation theory of the Mickelsson algebra describes, in some sense, the representation theory of A_n with its connections to the representation theory of $\mathrm{U}(\mathfrak{sl}(2))$ via the space of primitive vectors, which we realized as the solution space to the Laplacian. References and the general framework are given after the following motivational questions for my work in this topic.

Question II.1 (Presentation and automorphisms). *How does one construct the reduction superalgebra of a Lie superalgebra—more precisely, what are the generators and relations of these superalgebras that are not presented in the literature? What automorphisms can we construct?*

Question II.2 (Center of superalgebra). *Given that any particular reduction superalgebra of a Lie superalgebra is an associative superalgebra, can we fully characterize its (ghost) center?*

Question II.3 (Representation theory). *Can we classify the finite-dimensional irreducible representations of a reduction superalgebra? Can we recover the representation theoretic results of the underlying Lie superalgebra?*

Question II.4 (Physical applications). *What governs the connection between reduction algebras and integrable systems and R-matrix formalism, including finding solutions to the dynamical Yang-Baxter equation?*

i) From Mickelsson to Hartwig–Williams

The history of Mickelsson algebras is based in the "reduction problem" of decomposing representations of an associative algebra containing the universal enveloping algebra of a reductive Lie algebra [40, 46, 47], especially when the original module was an irreducible module over the parent algebra. It was conjectured by Mickelsson and proved by van den Hombergh that irreducible modules of $\mathcal{A}$ correlate with irreducible modules of Z: The Mickelsson algebra acts irreducibly in the space of primitive vectors of an irreducible $\mathcal{A}$-module.

Generalizing the example above, let $\mathcal{A} \supset U(\mathfrak{g})$ be an associative algebra containing the universal enveloping algebra $U(\mathfrak{g})$ of a reductive Lie algebra $\mathfrak{g}$. Denote by I the left ideal $\mathcal{A}\mathfrak{g}_+$ in $\mathcal{A}$ generated by the positive nilpotent part $\mathfrak{g}_+$ from the fixed triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then $N_{\mathcal{A}}(I)/I$, where $N = N_{\mathcal{A}}(I) = \{a \in \mathcal{A} \mid \mathfrak{g}_+ a \subset I\}$, is the subquotient of $\mathcal{A}$ termed the *Mickelsson algebra.* The Mickelsson algebra is equivalent/isomorphic[24, to see detailed derivations] to the following:

1. The space of primitive vectors $(\mathcal{A}/I)^+$ of the universal highest weight (left) module $\mathcal{A}/I$ with respect to $\mathfrak{g}_+$, i.e., the $\mathfrak{g}_+$ invariants of $\mathcal{A}/I$
2. The opposite algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{A}/I)^{\mathrm{op}}$ of left $\mathcal{A}$-module endomorphisms on $\mathcal{A}/I$
3. The double coset space $\mathcal{A}/\mathrm{I\!I} = J \backslash \mathcal{A}/I$ of $\mathfrak{g}_-$-coinvariants of $\mathcal{A}/I$ with J the right ideal $\mathfrak{g}_-\mathcal{A}$ and $\mathrm{I\!I} = J + I$

The algebra structure on N/I is inherited from $\mathcal{A}$; similarly, the algebra structure on $\mathrm{End}_{\mathcal{A}}(\mathcal{A}/I)^{\mathrm{op}}$ is natural for sets of module endomorphisms. Through the extremal project [3, 44, 45], one can define an associative product $\Diamond$ [24, 31] on the double coset space $\mathcal{A}/\mathrm{I\!I}$. The study of Mickelsson algebras was furthered by articles [50, 51, 52] of Zhelobenko and the book [53] written in Russian. Zhelobenko showed that a localization of the Mickelsson algebra over the ring of so-called dynamical scalars [25, usage explained in the introduction] yielded a finitely generated algebra that others later termed the *Zhelobenko algebra* Z. More precisely, the algebra Z is the *D-localized reduction algebra* where D is a multiplicative set of $U(\mathfrak{h}) \setminus \{0\}$. Reduction algebras have found applications in both the quantum [4] and super setting[39] as generalizations of the constructions have been found. One particular reduction algebra is the diagonal reduction algebra [30, 32, 33], which is formed from the diagonal embedding of $\mathfrak{g}$ into $\mathfrak{g} \times \mathfrak{g}$ so that we can take $U(\mathfrak{g} \times \mathfrak{g}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{g})$ as an associative algebra containing a copy of $U(\mathfrak{g})$.

With Hartwig, I produced the first known works on the diagonal reduction superalgebra of the Lie superalgebra $\mathfrak{osp}(1|2)$ [24, 25] and potentially the only modern treatment of the structure and representation theory of reduction algebras in the super setting, which had been missing from the literature outside of proof-less acknowledgments of their existence.

ii) Results: The diagonal reduction algebra of $\mathfrak{osp}(1|2)$ and its representations

In the first paper [24], we construct the diagonal reduction algebra $Z = Z(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g}, D)$, with $\mathfrak{g} = \mathfrak{osp}(1|2)$ and D the multiplicative set generated by integer shifts of the one-element basis of the Cartan subalgebra $\mathfrak{h}$ of the diagonal image of $\mathfrak{osp}(1|2)$. We give a complete presentation of Z in terms of generators and relations and re-establish the extremal projector of $\mathfrak{osp}(1|2)$. Additional computations produce conversions between relations on N/I and $\mathcal{A}/\mathrm{II}$ with the diamond product $\Diamond$, where (*loc. cit. U*) $\mathcal{A} = D^{-1}U(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2)) = D^{-1}U(\mathfrak{h}) \otimes_{U(\mathfrak{h})} U(\mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2))$. We prove that Z has a PBW-type basis and leverage it often in the sequel [25] and to guide our upcoming work. As an application of the relations, we write down an infinite subgroup of automorphisms: The group of what we call dilation automorphisms is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{C}^*$.

The importance of the results in the first paper is in the demonstration that the original goals of Mickelsson carry forward to the super setting. Furthermore, reduction algebras themselves have documented [30, 34] connections to the reflection equation, the Arnaudon-Buffenoir-Ragoucy-Roche (ABBR) equation, and their solutions' ties to integrable systems in statistical mechanics [12, 13] through the dynamical Yang-Baxter equation.

Furthermore, the algebras have an interesting representation theory as seen in the example above. We classify all finite-dimensional irreducible representations of Z and give more infinite-dimensional irreducible representations. In doing so, we construct common objects in the representation theory of Lie superalgebras and associative superalgebras but now with the added complexity of the dynamical scalars:

1. We define Verma modules for Z.
2. We define Shapovalov forms for the Verma modules of Z.
3. We characterize the ghost center (the center plus anti-center [17]) of Z as a quotient of a polynomial algebra.

Our work answers completely or partially, in the case of Z, Questions II.1-II.3. In terms of the long-term advancement of the field, our systematic study of Z opens up pathways to investigate other type of reduction algebras and diagonal reduction algebras for the rest of the basic and strange Lie superalgebras, in the terms of the classification[28] given by Kac.

iii) Future: Reduction superalgebras of other Lie superalgebras

There are natural collaborative opportunities to studying reduction superalgebras based on my current work. Here are some open problems.

1. Construct the diagonal reduction algebra for higher rank orthosymplectic Lie superalgebras. Present generators and relations and automorphisms.
2. Construct the diagonal reduction algebra for low rank Lie superalgebras such as the strange Lie superalgebras $\mathfrak{p}(2), \mathfrak{q}(2)$ of periplectic and queer type, respectively. Present generators and relations.
3. Classify the irreducible representations of other reduction superalgebras using the techniques [25] in dealing with dynamical scalars.

4. Strong physics interest: Use reduction superalgebras to describe solutions to Maxwell's equations, which have interplay [38] with $\mathfrak{osp}(2|2)$.

5. Strong physics interest: Question II.4.

III Generalized Parking Functions

For a survey on parking functions, [6] is an excellent choice. It also recommended to read the exposition[19, 20] by Kimberly P. Hadaway.

i) What are parking functions?

In brief, parking functions are combinatorial objects that were introduced and enumerated [35] to explore hashing function in computer science. Since the 1960s, there have been many generalizations and appearances of parking functions, especially noting their tie to the Catalan numbers [2, 43] and other ubiquitous combinatorial objects. As the Postdoctoral Research Advisor of MSRI's Undergraduate Program (MSRI-UP) 2021, I advised undergraduates in research projects investigating defective parking functions, interval parking functions, and other generalized parking functions. Each group submitted a report and further work has continued for publication.

ii) Enumerating generalized parking functions

I have worked with groups of people at various academic/professional stages on several topics concerned with parking functions and have the pleasure of being co-author to five (5) undergraduate students, two (2) graduate students, and one (1) tenured professor in the area. It is a privilege and joy to advise students and use my position to affirm a practice of People over Math: I have the responsibility to cultivate the space for students, people, to actually do mathematics. Thus I am not surprised by the scholarly outcomes, which include an article [1] accepted in The American Mathematical Monthly, highlighting a commitment, execution, and production born from mentoring and advising centered in humanity. In [1], we give an explicit bijection between ideal states of the Tower of Hanoi game and parking functions which only allow one "bump" during the parking configuration determination. This result includes an enumerative finding related to the Lah numbers, which Lah introduced in *A new kind of numbers and its application in the actuarial mathematics* [36]. Such an example can serve as a bridge for interdisciplinary exploration for, say, actuarial students to conduct research in combinatorics.

Most recently, I wrote scripts in MATLAB to help with computational verification of our characterization of the parking function generalization that extends the work of [21] to objects such as barred preferential arrangements and higher Fubini numbers. The relevant article will enter review shortly and features several student co-authors, as well.

iii) Utilization of algebraic techniques

The world of parking functions is vast. I have special interest in exploring the representation theoretic facets such as ties to invariant theory attributed to Haiman [22] and extensions to the work put into reports by the MSRI-UP 2021 researchers. Currently, there are multiple papers forthcoming from the summer of 2021. Even better, each project yielded more opportunities and open problems for potential students and my advising of them in exploring combinatorial and algebraic research.

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