Statement on research

Briefly, my research concerns the algebraic underpinnings of supermathematics in the spirit of Felix Berezin. In particular, my current work involves the infinite-dimensional representation theory of the complex orthosymplectic Lie superalgebra $B(0|n)$:[1] orthosymplectic representation theory interests both mathematicians and physicists considering duality, as described by Howe-Schur-Sergeev, categorification, harmonic analysis, combinatorics, geometry, superstring theory, supersymmetry, and any number of subdisciplines with a super analogue. The present exposition provides historical significance of the research area broadly, results and contributions of the author, and future directions along with a call for collaboration.

1 History

1.1 Superalgebras

In 1941, Whitehead defined a product on the graded homotopy groups of a pointed topological space, the first non-trivial example of a Lie superalgebra. Whitehead's work in algebraic topology [Whi41] was known to mathematicians who defined new graded geo-algebraic structures, such as $\mathbb{Z}_2$-graded algebras, or superalgebras to physicists, and their modules. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ would come to be a $\mathbb{Z}_2$-graded vector space, even (respectively, odd) elements found in $g_{\bar{0}}$ (respectively, $\mathfrak{g}_{\bar{1}}$), with a parity-respecting bilinear multiplication termed the Lie superbracket $[\cdot,\cdot]$ inducing[2] a symmetric intertwining map $\mathfrak{g}_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ of $(\mathfrak{g}_{\bar{0}}, [\cdot,\cdot]\big|_{\mathfrak{g}_{\bar{0}}})$-modules. Researchers of the 1960s and 1970s furthered the systematic study of Lie superalgebras with a view for supermanifolds and the need for physicists to extend the classical symmetry formulation behind Wigner's Nobel Prize to one incorporating bosonic ("even") and fermionic ("odd") particles simultaneously—the famed supersymmetry (SUSY) [see FF77; Zum06]. In many regards, supermathematics is synonymous with the study of $\mathbb{Z}_2$-graded structures and spaces with Grassmann-valued coordinates. Now the algebraic development of supermathematics has become its own source of motivation—even providing a dictionary [FSS00]—beginning with the classification of simple finite-dimensional Lie superalgebras over algebraically closed fields of characteristic zero in [Kac75].

1.2 Orthosymplectic Lie Superalgebras

Still, the investigation of $\mathbb{Z}_2$-graded non-associative algebras featuring orthogonal and symplectic parts, the so-called orthosymplectic Lie superalgebras, proves to be quite fruitful within and outside of mathematics. Specifically, the Lie superalgebra $\mathfrak{osp}(1|2n,\mathbb{R})$[1] is of great importance to superconformal theories; see chapters 6 and 7 of [Far84] for an early survey of physical applications. In mathematics, the natural question to describe simple objects in a module category of $\mathfrak{osp}(1|2n,\mathbb{C})$, and of other classical Lie superalgebras defined in [Kac77], was considered by Dimitrov, Mathieu, Penkov in [DMP00]. Gorelik and Grantcharov completed the classification began in [DMP00] by publishing [FGG16] then [GG18], the former with Ferguson, who classified the simple bounded highest-weight modules of $\mathfrak{osp}(1|2n,\mathbb{C})$ in his thesis [Fer15]. Primitive vectors in certain tensor product representations of $\mathfrak{osp}(1|2n,\mathbb{C})$ were essential to the completion of [Fer15]. Coulembier also paid close attention to primitive vectors in [Cou13] to determine whether tensor products of certain irreducible highest-weight $\mathfrak{osp}(2m+1|2n,\mathbb{R})$-representations were completely reducible; a series of papers by Coulembier and co-authors [Cou10; Cou13; CSS14] motivated inspecting the reducibility of certain $\mathfrak{osp}(1|2n,\mathbb{R})$-modules to establishing a

[1] The series of Lie superalgebras $B(m|n)$ over a field $\mathbb{K}$ of characteristic zero is the series of orthosymplectic algebras $\mathfrak{osp}(2m+1|2n,\mathbb{K})$.

[2] We also require $[x,[x,x]] = 0, \ \forall x \in \mathfrak{g}_{\bar{1}}$.

Clifford analysis on supermanifolds. The case of $m = 0$, again, $\mathfrak{osp}(1|2n)$, serves to guide a different exploration described below.

2 Contributions

2.1 Specific Problems

Henceforth, the ground field is $\mathbb{C}$ and $\mathfrak{g} = \mathfrak{osp}(1|2n)$. Consider $\mathbb{C}^{1+2n}$ as a $\mathbb{Z}_2$-graded vector space (super vector space):

$$\mathbb{C}^{1|2n} = (\mathbb{C}^{1|2n})_{\bar{0}} \oplus (\mathbb{C}^{1|2n})_{\bar{1}} = \mathbb{C}v_0 \oplus \left(\bigoplus_{i>0} \mathbb{C}v_i \right),$$

where $v_i = e_{i+1}$, $0 \leq i \leq 2n$, are the standard basis vectors of $\mathbb{C}^{1+2n}$. Define the Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(1|2n)$ as the set of endomorphisms on $\mathbb{C}^{1|2n}$ preserving the form represented by the matrix

$$J_n = \overbrace{\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & -I_n \\ 0 & I_n & 0 \end{array}\right]}^{(1+2n)\times(1+2n)}$$

with the Lie superbracket on any pair of homogeneous morphisms[3] X and Y:

$$[X, Y] = XY - (-1)^{|X||Y|} YX.$$

Main goal: to decompose the $\mathfrak{g}$*-module* $\mathbb{C}[x_1, x_2, \ldots, x_n] \otimes \mathbb{C}^{1|2n}$ *into the direct sum of irreducible submodules.*

The following two facts play a crucial role in pursuing our goal. First, $\mathfrak{g}$ is generated by odd root vectors $\{X_{\pm\delta_i}\}_{i=1}^n$. Secondly, $\mathfrak{g}$ acts on $\mathbb{C}[x_1, x_2, \ldots x_n]$ via the superalgebra epimorphism $\mathcal{U}(\mathfrak{g}) \twoheadrightarrow \mathcal{D}(n)$ mapping the universal enveloping algebra of $\mathfrak{g}$ onto the n^{th} Weyl (super)algebra generated by $\{x_i, \partial_{x_i}\}_{i=1}^n$, as communicated in [Fer15]. Then $\mathfrak{g}$ acting on the tensor product representation $\mathbb{C}[x_1, x_2, \ldots, x_n] \otimes \mathbb{C}^{1|2n}$ sparks a classical inquiry into the wide-open infinite-dimensional representation theory of the Lie superalgebra $\mathfrak{g}$:

(i) Does the infinite-dimensional tensor product representation $\mathbb{C}[x_1, x_2, \ldots, x_n] \otimes \mathbb{C}^{1|2n}$ decompose into a direct sum of simple $\mathfrak{g}$-submodules?

(ii) Are there formulas giving explicit realizations of $\mathfrak{g}$ acting on the submodules found in (i)?

2.2 Results and Techniques Used

My work has answered both questions (i) and (ii) in the affirmative. Acting on a known primitive vector from [Fer15] gives rise to a "fake Casimir" element called S, a quadratic element of $\mathcal{U}(\mathfrak{g})$, and operators $\Gamma^- = 1 - 2S$ and $\Gamma^+ = 1 + 2S$. Then Theorems 1 and 2 answer (i) and Corollaries 3 and 4 answer (ii).

Theorem 1. *[GW] The operators* Γ^- *and* Γ^+ *are (super) vector space automorphisms on*

$$V = \mathbb{C}[x_1, x_2, \ldots, x_n] \otimes \mathbb{C}^{1|2n}.$$

[3] Every pair of objects in the category of complex super vector spaces $\mathbf{SVect}_{\mathbb{C}}$ has an internal Hom; that is, there is a natural grading on morphisms corresponding to parity-preserving and parity-reversing maps.

Theorem 2. *[GW] The $\mathfrak{g}$-module V is a direct sum of two simple submodules:*

$$V = \Gamma^-(V^0) \oplus \Gamma^+(V^+),$$

where

$$V^0 = \mathbb{C}[x_1, x_2, \ldots, x_n] \otimes \mathbb{C}v_0,$$
$$V^+ = \mathbb{C}[x_1, x_2, \ldots, x_n] \otimes (\bigoplus_{i>0} \mathbb{C}v_i).$$

Let $M^0 = \Gamma^-(V^0)$ and $M^+ = \Gamma^+(V^+)$. Then M^0 is the only simple infinite-dimensional completely-pointed $\mathfrak{g}$-module (up to isomorphism; [see Cou13, Section 6]). Moreover, if $\boldsymbol{x}^{\boldsymbol{k}} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$, $1 \leq i \leq 2n$, then $\{\Gamma^-(\boldsymbol{x}^{\boldsymbol{k}} \otimes v_0)\}_{\boldsymbol{k}}$ and $\{\Gamma^+(\boldsymbol{x}^{\boldsymbol{k}} \otimes v_i)\}_{\boldsymbol{k},i}$ form bases of M^0 and M^+, respectively.

Note, all $\mathfrak{g}$-modules considered in this section are weight modules, i.e., decompose into the direct sum of weight spaces. In fact, the finite-dimensional representations of $\mathfrak{g}$ are semisimple. We remind the reader that the results of this section are part of infinite-dimensional representation theory. The ensuing corollaries describe the action of $\mathfrak{g}$ on M^0 and on M^+ by viewing Γ^- as a conjugating element inside the (super)algebra $\mathrm{End}(V^0)$ and Γ^+ as a conjugating element inside the (super)algebra $\mathrm{End}(V^+)$.

Corollary 3. *[GW] The following formulas hold on V^0:*

$$(\Gamma^-)^{-1} X_{\pm\delta_j} \Gamma^- = -X^{(1)}_{\pm j},\ 1 \leq j \leq n;$$

$X^{(1)}_{\pm j}$ is defined by

$$X^{(1)}_j = X_{\delta_j} \otimes \mathbb{1},$$
$$X^{(1)}_{-j} = X_{\delta_{-j}} \otimes \mathbb{1}.$$

Corollary 4. *[GW] The following formulas hold on V^+:*

$$(\Gamma^+)^{-1} X_{\pm\delta_j} \Gamma^+ = \Psi_{\pm j},\ 1 \leq j \leq n;$$

$\Psi_{\pm j}$ is defined by

$$\Psi_j = X_{\delta_j} \otimes \mathbb{1} - \sqrt{2} \sum_{l=1}^n x_l \otimes E_{jl} + \sqrt{2} \sum_{l=1}^n \partial_{x_l} \otimes E_{j,n+l},$$
$$\Psi_{-j} = X_{-\delta_j} \otimes \mathbb{1} + \sqrt{2} \sum_{l=1}^n x_l \otimes E_{n+j,l} - \sqrt{2} \sum_{l=1}^n \partial_{x_l} \otimes E_{n+j,n+l};$$

and, $E_{pq} v_r = \delta_{qr} v_p$.

2.3 Conjectures

There is strong evidence to suggest a $\mathfrak{g}$-module structure on the space of polynomials in n commuting variables and $2n$ anti-commuting variables,

$$N = \mathbb{C}[x_1, x_2, \ldots, x_n] \otimes \mathbb{C}[\xi_1, \xi_2, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{2n}] \cong Sym(\mathbb{C}^n) \otimes \bigwedge \mathbb{C}^{2n},$$

through the polarization[4] of $\Psi_{\pm j}$. Of interest is the generalization of the operators S, Γ^-, and Γ^+ to achieve a decomposition of N and to illuminate formulas, primitive vectors, and bases.

3 Future Directions

3.1 Thesis and Beyond

My interest in supermathematics intersects with geometric pursuits, ties to traditional themes within noncommutative algebra, and applies to the combinatorial, graph theoretical, and physical expressions of SUSY. In geometry, the suspected $\mathfrak{osp}(1|2n)$-module N of Subsection 2.3 is identified with the set of super polynomials $\mathcal{P}$ sitting inside the structure sheaf on superspace $\mathbb{C}^{n|2n}$, as in [Cou10]:

$$\mathcal{P} = \mathbb{C}[x_1, x_2, \dots, x_n] \otimes \bigwedge \mathbb{C}^{2n} \subset \mathcal{O}(\mathbb{C}^{n|2n}) = \mathcal{C}^\infty(\mathbb{C}^n) \otimes \bigwedge \mathbb{C}^{2n}.$$

Through the lens of invariant theory, the $\mathfrak{gl}(n)$-module structure of N hearkens to the language of Howe in [How89]. To the graph theoretic claim, I point to the literature on Adinkras found in the references of [Goi+19] and [Zha14] and the need for translations to assist mathematicians[5] and physicists studying representations of SUSY algebras. Lastly, there is a desired categorical viewpoint to illustrate the results in a functorial manner. At least one aspect of this discussion constitutes ongoing work towards my thesis and other parts are more or less near-term goals.

3.2 Potential Collaborations

The techniques utilized in obtaining the results of Subsection 2.2 are transferable to many areas of mathematical research as representation theory and Lie-theoretic notions have a long history of connecting geometry, differential equations, combinatorics, algebra, etc. There are more (super)algebras with rich module theory to explain, perhaps through differential operators and morphisms of associative (super)algebras. As seen above, algebraic perspectives can add to the overall body of mathematical research through various pathways. Beyond the discussion of Subsection 3.1, I seek challenging problems and productive collaboration along the career arc of a mathematician.

3.3 Broader Impact

I also work to enrich research-oriented careers of mathematicians whose representation theory is scarce relative to mentorship. As a field, mathematics has a long history of apprenticeship—a topic of research articles [MOA10, e.g.]—and with increased efforts in making higher mathematics a feasible option of study, there is a greater need for understanding and shaping the modern culture of mathematics to foster mathematical discovery. It is with this light that I have research interests in the bibliographical collections of mathematicians and the geometry of research output—the who, the what, the when, the where, the why, and, certainly, the how.

---

[4] I am grateful for Markus Hunziker sharing the terminology with me.
[5] I am grateful for Edray Goins introducing me to Adinkras.

References

[Cou10] Coulembier, K. (2010). The orthosymplectic lie supergroup in harmonic analysis. In *Aip conference proceedings* (Vol. 1281, pp. 1459–1463). doi:10.1063/1.3498035. arXiv: 1202.0668

[Cou13] Coulembier, K. (2013). On a class of tensor product representations for the orthosymplectic superalgebra. *Journal of Pure and Applied Algebra*, *217*(5), 819–837. doi:10.1016/j.jpaa.2012.09.009. arXiv: 1205.0119

[CSS14] Coulembier, K., Somberg, P., & Souček, V. (2014). Joseph ideals and harmonic analysis for $\mathfrak{osp}(m|2n)$. *International Mathematics Research Notices*, *2014*(15), 4291–4340. doi:10.1093/imrn/rnt074. arXiv: 1210.3507

[DMP00] Dimitrov, I., Mathieu, O., & Penkov, I. (2000). On the structure of weight modules. *Transactions of the American Mathematical Society*, *352*(6), 2857–2869. doi:10.1090/S0002-9947-00-02390-4

[Far84] Farmer, R. (1984-10). *Orthosymplectic superalgebras in mathemathics and physics* (Doctoral dissertation, Tasmania U.). Retrieved from http://eprints.utas.edu.au/19542/1/whole_FarmerRichardJoseph1985_thesis.pdf

[FF77] Fayet, P., & Ferrara, S. (1977). Supersymmetry. *Physics Reports. A Review Section of Physics Letters*, *32C*(5), 249–334. doi:10.1016/0370-1573(77)90066-7

[Fer15] Ferguson, T. (2015). *Weight Modules of Orthosymplectic Lie Superalgebras* (Doctoral dissertation, The University of Texasa at Arlington).

[FGG16] Ferguson, T., Gorelik, M., & Grantcharov, D. (2016). Bounded highest weight modules over $\mathfrak{osp}(1, 2n)$. In *Lie algebras, Lie superalgebras, vertex algebras and related topics* (Vol. 92, pp. 135–144). Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI.

[FSS00] Frappat, L., Sciarrino, A., & Sorba, P. (2000). *Dictionary on Lie algebras and superalgebras*. With 1 CD-ROM (Windows, Macintosh and UNIX). Academic Press, Inc., San Diego, CA.

[Goi+19] Goins, E., Iga, K., Kostiuk, J., & Stiffler, K. (2019). The signed monodromy group of an adinkra. arXiv: http://arxiv.org/abs/1909.02609v1 [math.CO]

[GG18] Gorelik, M., & Grantcharov, D. (2018). Simple bounded highest weight modules of basic classical lie superalgebras. arXiv: http://arxiv.org/abs/1812.11593v1 [math.RT]

[GW] Grantcharov, D., & Williams, D., II. *Bases of infinite-dimensional tensor product representations of* $\mathfrak{osp}(1|2n)$. In preparation.

[How89] Howe, R. (1989). Remarks on classical invariant theory. *Transactions of the American Mathematical Society*, *313*(2), 539–570. doi:10.2307/2001418

[Kac75] Kac, V. (1975). Classification of simple Lie superalgebras. *Akademija Nauk SSSR. Funkcionalnyi Analiz i ego Priloženija*, *9*(3), 91–92.

[Kac77] Kac, V. (1977). Lie superalgebras. *Advances in mathematics*, *26*(1), 8–96.

[MOA10] Malmgren, R., Ottino, J., & Amaral, L. (2010). The role of mentorship in protege performance. *Nature 465: 622-626 (2010)*. doi:`10.1038/nature09040`. arXiv: `http://arxiv.org/abs/1006.2312v1 [physics.soc-ph]`

[Whi41] Whitehead, J. (1941). On Adding Relations to Homotopy Groups. *The Annals of Mathematics*, *42*(2), 409. doi:`10.2307/1968907`

[Zha14] Zhang, Y. (2014). Adinkras for mathematicians. *Transactions of the American Mathematical Society*, *366*(6), 3325–3355. doi:`10.1090/S0002-9947-2014-06031-5`

[Zum06] Zumino, B. (2006). Supersymmetry then and now. *Fortschritte der Physik. Progress of Physics*, *54*(2-3), 199–204. doi:`10.1002/prop.200510268`