The diagonal reduction superalgebra of $\mathfrak{osp}(1|2)$ and its representations

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In [HW22], the diagonal reduction algebra $Z(\mathfrak{G}, \mathfrak{g}; D)$ of the Lie superalgebra $\mathfrak{osp}(1|2)$ is initially given as a quotient algebra isomoprhic to the superalgebra A

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- $\mathfrak{H}$: Cartan sublagebra of $\mathfrak{g}$: $\quad \mathbb{C}H$
- D: $\langle \{H + n \mid n \in \mathbb{Z}\} \rangle_{\text{monoid}}$ multipicative set

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$$[h, x_{\pm k\alpha}] = \mp k x_{\pm k\alpha}, \qquad [x_{-k\alpha}, x_{k\alpha}] = h, \;\; k \in \{1,2\},$$
$$[x_{\pm\alpha}, x_{\pm\alpha}] = \mp 2 x_{\pm 2\alpha}, \qquad [x_{\pm\alpha}, x_{\mp 2\alpha}] = x_{\mp\alpha}, \qquad [x_{\pm 2\alpha}, x_{\pm\alpha}] = 0.$$

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- triangular decomposition:

$$\mathfrak{osp}(1|2) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad \mathfrak{h} = \mathbb{C}h, \quad \mathfrak{n}_\pm = \mathbb{C}x_{\pm 2\alpha} \oplus \mathbb{C}x_{\pm\alpha}$$

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- $U = R \otimes_{U(\mathfrak{H})} U(\mathfrak{G})$
- $I = U\mathfrak{N}_+$
- $Z = Z(\mathfrak{G}, \mathfrak{g}; D) = N_U(I)/I$ $\quad$ $N_U(I)$ is the normalizer of I in U

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- $\lozenge$ is an associative product on the double coset space $U/\mathrm{I\!I}$ defined through the extremal projector [Tol85; Tol11; HW22].

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- $\Diamond$ is an associative product on the double coset space $U/\mathrm{I\!I}$ defined through the extremal projector [Tol85; Tol11; HW22].
- Generators of the reduction algebra A (as an R-ring): $\bar{x}_\beta = \widetilde{x}_\beta + \mathrm{I\!I}$, $\bar{h} = \widetilde{h} + \mathrm{I\!I}$

For a more thorough account of superified spaces, their maps, and other non-classical notions: [CW12; Mus12] or Section 2-b of [BK02].

References

[BK02] J. Brundan and A. Kleshchev. "Hecke-Clifford superalgebras, crystals of type $A_{2l}^{(2)}$ and modular branching rules for $\widehat{S}_n$". *arXiv:math/0103060* (July 2002). arXiv: math/0103060.

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